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On the Analytic Continuation of Functions Which Map the Upper Half Plane into Itself

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I. INTRODUCTION

Let $f(z)$ be analytic in the upper half plane and map the upper half plane into itself. Then it is known [3, p. 25] that $f(z)$ is of the form

$$f(z) = Az + b + \int_{-\infty}^{\infty} \frac{1 + tz}{t - z} \, d\psi(t) \quad (\operatorname{Im} z > 0), \tag{1.1}$$

where $\psi(t)$ is a bounded non-decreasing function (i.e., a mass distribution), $A \geqslant 0$, and b is real. More explicity,[1]

$$A = \lim_{y \to \infty} \frac{f(iy)}{iy} = \operatorname{Im} f(i) - \int_{-\infty}^{\infty} d\psi(t) \tag{1.2}$$

and

$$b = \operatorname{Re} f(i). \tag{1.3}$$

If in addition, $iyf(iy) = 0(1)$ as $y \to \infty$, then it is further known [3, p. 25] that

$$f(z) = \int_{-\infty}^{\infty} \frac{d\gamma(t)}{t - z} \quad (\operatorname{Im} z > 0), \tag{1.4}$$

[1] The second part of formula (1.2) and formula (1.3), while trivial to derive, do not appear in the literature.

where $d\gamma(t)$ is another mass distribution related to $d\psi(t)$ by $d\gamma(t) = (1 + t^2)d\psi(t)$ (thus the boundedness condition on $f(z)$ guarantees that $d\psi(t)$ has a second moment).

This paper is concerned with the analytic continuation of $f(z)$ given by (1.1) or (1.4) into the lower half plane. Such continuation is easily done across any interval (a,b) in which $d\psi(t)$ or $d\gamma(t)$ has no mass. For inspection of (1.1) or (1.4) shows that in such intervals these formulas define $f(z)$ to be real analytic, so that $f(z)$ may be continued across (a,b) by reflection.

Of more interest is the problem of obtaining non-reflective continuations. In particular, recent work of Dolph and Penzlin [ref. 1, p. 14] shows the desirability of obtaining such continuations of (1.4) in order to study the complex spectra of certain non-self-adjoint operators.

Our principal theorem is the following:

THEOREM 1.1. *If $f(z)$ is given by (1.1), then it can be continued across (a,b) into the lower half plane if and only if $\psi(t)$ is real analytic in (a,b). If $\psi(t)$ is real analytic in (a,b) and if $\psi(z)$ is the analytic extension of $\psi(t)$ as defined in (a,b), then the continuation of $f(z)$ across (a,b) is given by*

$$f(z) = \overline{f(\bar{z})} + 2\pi i(1 + z^2)\psi'(z). \tag{1.5}$$

An immediate corollary of Theorem 1.1 is the following theorem, stated here because of its possible applications in operator theory.

THEOREM 1.2. *If $f(z)$ is given by (1.4), then $f(z)$ can be continued across (a,b) into the lower half plane if and only if $\gamma(t)$ is real analytic in (a,b). If $\gamma(t)$ is real analytic in (a,b), then the continuation of $f(z)$ across (a,b) is given by*

$$f(z) = \overline{f(\bar{z})} + 2\pi i\gamma'(z). \tag{1.6}$$

A proof of Theorem 1.2 independent of Theorem 1.1 can be based on the Stieltjes inversion formula

$$\gamma(t_2) - \gamma(t_1) = \lim_{y\to 0+} \frac{1}{\pi}\int_{t_1}^{t_2} \operatorname{Im} f(x + iy)\, dx$$

(A proof of this inversion formula may be found in ref. 4, p. 163.) Such a proof is quite similar to the proof which we shall give for Theorem 1.1, based on the inversion formula for (1.1) which we shall obtain in the next section of this paper.

II. An Inversion Formula for Functions which Map the Upper Half Plane into Itself[2]

Consider the integral

$$\int_{t_1}^{t_2} \operatorname{Im} f(x+iy)\,dx = Ay(t_2-t_1) + \int_{t_1}^{t_2} dx \int_{-\infty}^{\infty} \operatorname{Im}\left(\frac{1+tz}{t-z}\right) d\psi(t) \qquad (2.1)$$

Since

$$\frac{1+tz}{t-z} = z + \frac{1+z^2}{t-z} = -t + \frac{1+t^2}{t-z},$$

we see that for fixed positive y,

$$\left|\frac{1+t(x+iy)}{t-x-iy}\right|$$

is uniformly bounded for all real t and for all $x \in [t_1,t_2]$, so that Fubini's theorem may be applied to obtain

$$\int_{t_1}^{t_2} \operatorname{Im} f(x+iy)\,dx = Ay(t_2-t_1) + \int_{-\infty}^{\infty} (1+t^2)\,d\psi(t)\operatorname{Im}\left(\int_{t_1}^{t_2} \frac{dx}{t-x-iy}\right)$$

$$= Ay(t_2-t_1) + \int_{-\infty}^{\infty} (1+t^2)\Theta_y(t)\,d\psi(t), \qquad (2.2)$$

where $\Theta_y(t)$ is the angle subtended at t by t_1+iy and t_2+iy.

It is readily verified that, for fixed t, $\Theta_y(t)$ is a monotone function of y, the type of monotonicity depending on whether or not $t \in [t_1, t_2]$. Furthermore,

$$\lim_{y\to 0^+} \Theta_y(t) = \begin{cases} \pi & t \in (t_1,t_2) \\ \dfrac{\pi}{2} & t = t_1 \quad \text{or} \quad t_2 \\ 0 & \text{otherwise.} \end{cases}$$

[2] I have recently learned that my formula has also been obtained by F. Nevanlinna and T. Nieminen [2, p. 14]. I nevertheless believe my method of proof to be of interest. Furthermore its inclusion makes the present paper more self-contained.

Writing the last integral in (2.2) as

$$\int_{-\infty}^{t_1-0} + \int_{t_1-0}^{t_1+0} + \int_{t_1+0}^{t_2-0} + \int_{t_2-0}^{t_2+0} + \int_{t_2+0}^{\infty},$$

we may apply monotone convergence theorems to these integrals to obtain

$$\lim_{y\to 0^+} \int_{t_1}^{t_2} \operatorname{Im} f(x+iy)\, dx = \int_{t_1}^{t_2} (1+t^2)\, d\psi(t),$$

provided that $\psi(t)$ has been given the normalization

$$\psi(t) = \tfrac{1}{2}(\psi(t+0) + \psi(t-0))\,.^3$$

Note that we could not have used the dominated convergence theorem since $d\psi(t)$ may fail to have a second moment. Thus we have the following theorem:

THEOREM 2.1. Let $f(z)$ be given by (1.1) and let $\gamma(t) = \int_0^t (1+x^2)\, d\psi(x)$. Then

$$\gamma(t_2) - \gamma(t_1) = \lim_{y\to 0^+} \frac{1}{\pi} \int_{t_1}^{t_2} \operatorname{Im} f(x+iy)\, dx \tag{2.3}$$

and

$$\psi(t_2) - \psi(t_1) = \int_{t_1}^{t_2} \frac{d\gamma(t)}{1+t^2}\,. \tag{2.4}$$

The following examples illustrate the inversion formulas (2.3) and (2.4):

(i) The most trivial example of an analytic function mapping the upper half plane into itself is $f(z) = i$. From formulas (1.2) and (1.3), we

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[3] Such normalization is equivalent to redistributing the mass so that $\int_{t_0+0}^{t_0-0} d\psi(t) = 2\int_{t_0-0}^{t_0} d\psi(t) = 2\int_{t_0}^{t_0+0} d\psi(t)$. The full significance of this redistribution is perhaps best seen by recalling that any Lebesgue-Stieltjes integral $\int_a^b f(t)\, d\psi(t)$ can be rewritten as the ordinary Lebesgue integral $\int_{\psi(a)}^{\psi(b)} f(t(\psi))\, d\psi$.

have $A = b = 0$. Since $\operatorname{Im} f(z) = 1$, it follows that $d\gamma(t) = 1/\pi\, dt$ and $d\psi(t) = dt/\pi(1 + t^2)$. Hence

$$i = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1 + tz}{t - z}\frac{dt}{1 + t^2} \quad (\operatorname{Im} z > 0),$$

which is easily verified by contour integration. Note that the value of the integral is equal to $-i$ for z in the lower half plane.

(ii) Let $f(z)$ be that branch of $z^{1/2}$ which is analytic in the plane with the non-positive real axis omitted and maps the positive real axis onto itself. Then it is easily seen that $\operatorname{Im} f(x + iy)$ tends continuously to

$$g(x) = \begin{cases} |x|^{1/2} & x < 0 \\ 0 & x \geqslant 0 \end{cases}$$

as $x + iy$ approaches the real axis from the upper half plane. Hence the mass is restricted to $(-\infty, 0)$ with $d\psi(t) = |t|^{1/2}\, dt/(\pi(1 + t^2))$ there. From (1.2) and (1.3), we have $A = 0$, $b = \sqrt{2}/2$. Hence

$$f(z) = \frac{\sqrt{2}}{2} + \frac{1}{\pi}\int_{-\infty}^{0} \frac{1 + tz}{t - z}\frac{|t|^{1/2}}{1 + t^2}\, dt \quad (\operatorname{Im} z > 0).$$

(iii) Let $f(z)$ be that branch of $z^{1/4}$ which is analytic in the plane with the non-positive real axis omitted and for which $\pi/4 < \arg f(z) < 3\pi/4$. In this case $A = 0$, $b = \cos 5\pi/8 = -\frac{1}{2}\sqrt{2 - \sqrt{2}}$, and $\operatorname{Im} f(x + iy)$ tends continuously to

$$g(x) = \begin{cases} \dfrac{\sqrt{2}}{2}|x|^{1/4} & x < 0 \\ x^{1/4} & x \geqslant 0 \end{cases}$$

as $x + iy$ approaches the real axis from the upper half plane. Hence

$$f(z) = -\frac{1}{2}\sqrt{2 - \sqrt{2}} + \frac{1}{\pi}\left[\frac{\sqrt{2}}{2}\int_{-\infty}^{0} \frac{1 + tz}{t - z}\frac{|t|^{1/4}}{1 + t^2}\, dt + \int_{0}^{\infty} \frac{1 + tz}{t - z}\frac{t^{1/4}}{1 + t^2}\, dt\right]$$

$$(\operatorname{Im} z > 0).$$

III. Proof of Principal Theorem

Suppose $f(z)$ as defined by (1.1) can be continued across (a, b), and let $[x_1, x_2] \subset (a, b)$. Then

$$\lim_{y \to 0^+} \operatorname{Im} f(x + iy) = \frac{1}{2i} (f(x) - \bar{f}(x)) = g(x)$$

uniformly in $[x_1, x_2]$, $\bar{f}(z)$ being the analytic function $\overline{f(\bar{z})}$. From the inversion formula (2.3), we conclude that

$$\int_{x_1}^{x_2} (1 + t^2)\, d\psi(t) = \frac{1}{\pi} \int_{x_1}^{x_2} g(x)\, dx.$$

Consequently $\psi(t)$ is absolutely continuous on (a, b) with $\psi'(t)$ equal to the real analytic function $g(t)/\pi(1 + t^2)$. Hence $\psi(t)$ is real analytic in (a,b).

Conversely, suppose $\psi(t)$ is real analytic in (a, b), and let $[x_1, x_2] \subset (a, b)$. Then we may write $f(z)$ as

$$\int_{t \notin [x_1, x_2]} \frac{1 + tz}{t - z}\, d\psi(t) + \int_{x_1}^{x_2} \frac{1 + tz}{t - z}\, \psi'(t)\, dt = I_1(z) + I_2(z)$$

Since $I_1(z)$ is real analytic on (x_1, x_2), we need only show that $I_2(z)$ can be continued across (x_1, x_2).

Since for fixed z in the upper half plane the integrand in $I_2(z)$ is analytic in t, $I_2(z)$ may be rewritten as

$$I_2(z) = \int_C \frac{1 + \tau z}{\tau - z}\, \psi'(\tau)\, d\tau,$$

where the path of integration C is a rectifiable arc with endpoints x_1 and x_2, all other points of which lie in the lower half of the complex τ-plane. This new integral representation of $I_2(z)$ gives an analytic continuation of $I_2(z)$ across (x_1,x_2) into the domain bounded by (x_1,x_2) and C. Thus $f(z)$ is analytic in (a,b).

To obtain the continuation formula (1.5), note that $f(z) - \pi i(1 + z^2)\psi'(z)$ is real analytic in (a,b) and may be continued across (a,b) by reflection. Thus we have

$$f(z) - \pi i(1 + z^2)\psi'(z) = \overline{f(\bar{z})} + \pi i(1 + z^2)\psi'(z).$$

IV. Remarks on Continuation and Some Examples

The continuation formulas (1.5) and (1.6) show that the behavior in the lower half plane of a given continuation of $f(z)$ depends on the analytic function $\gamma'(z) = (1 + z^2)\psi'(z)$ defined by the values of $\psi(t)$ in the interval of continuation. In particular, singularities in the lower half plane of any continuation are singularities of the pertinent $\gamma'(z)$.

If two disjoint intervals determine different $\gamma'(z)$, then (1.1) is the restriction to the upper half plane of a branch of a multiple valued analytic function. Any common endpoint of two such intervals is a branch point.[4]

Any limit point of points on the real axis at which $\psi(t)$ is not analytic will be a non-isolated singularity of $f(z)$. Thus by taking any $\psi(t)$ having jumps at all rational points, we can construct an $f(z)$ having the real axis as a natural boundary.

That poles can occur in the lower half plane can be seen by taking $\gamma'(t) = (1 + t^2)^{-n}$. To produce an essential singularity in the lower half plane, we can take $\gamma'(t) = \exp[- t^2 - (1 + t^2)^{-1}]$.

We close with some examples.

(i) Let $f(z)$ be the branch of $z^{1/2}$ considered in example (ii) of the preceding section. In this case there are two possible continuations, corresponding to the positive and the negative real axis. For the positive real axis, formula (1.5) yields continuation by reflection. To obtain the continuation $f_1(z)$ across the negative real axis, we must obtain a function analytic in the lower half plane and equal to $1/\pi\, |t|^{1/2}$ fot the negative t. The function $i/\pi\, f(z)$ meets the requirements. Since $\overline{f(\bar z)} = f(z)$, the continuation formula becomes $f_1(z) = f(z) - 2f(z) = - f(z)$, corresponding to the fact that continuation across the negative real axis takes us onto the other sheet of the Riemann surface of $z^{1/2}$.

(ii) Let $f(z)$ be the branch of $z^{1/4}$ considered in example (iii) of the preceding section. There are two possible continuations, $f_1(z)$ across the positive real axis and $f_2(z)$ across the negative real axis. Since $- if(z)$ is the branch of $z^{1/4}$ mapping the positive real axis into itself, it is easily seen that $\overline{f(\bar z)} = - f(z)$. The analytic function $t^{1/4}/\pi$ continues to the lower half plane as $- i/\pi\, f(z)$. Hence $f_1(z) = - f(z) + 2f(z) = f(z)$, as was to be expected. The analytic function determined by $\sqrt{2}/2\, |t|^{1/4}/\pi$ is given for $\operatorname{Im} z < 0$ by $e^{-i\pi/4}(\sqrt{2}/2)\, f(z)/\pi$. Hence $f_2(z) = - f(z) + ie^{-i\pi/4}\sqrt{2}f(z)$ $= f(z)(- 1 + i + 1) = if(z)$, in accordance with the fact that crossing

[4] Provided, of course, that the common endpoint is no worse than an isolated singularity of the appropriate γ' for each interval.

the negative real axis from above takes us onto the sheet of the Riemann surface of $z^{1/4}$ for which $3\pi/4 < \arg z^{1/4} < 5\pi/4$.

(iii) Consider

$$f(z) = \int_{-\infty}^{\infty} \frac{e^{-t^2}}{t - z}\, dt \qquad (\operatorname{Im} z > 0).$$

According to Theorem 1.2, this is the restriction to the upper half plane of an entire function. Indeed, by use of Fourier and Laplace integrals it is easily shown that

$$f(z) = \pi i e^{-z^2} - 2\sqrt{\pi}\, e^{-z^2} \int_0^z e^{v^2}\, dv,$$

from which it is easy to verify that the relationship $f(z) = \overline{f(\bar{z})} + 2\pi i e^{-z^2}$ given by (1.6) does hold.

Note Added in Proof: As for isolated singularities on the real axis, it is easily seen that any poles of $f(z)$ on the real axis must be simple poles. That essential singularities can occur on the real axis is shown by taking $d\gamma(t)$ in (1.4) to be $\exp(-t^2 - t^{-2})$ and considering z equal to zero.

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